On Exact Solutions of a Coupled Korteweg – de Vries System

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The analytical positons, negatons, and complexitons and their interaction solutions to the coupled Korteweg – de Vries (KdV) system are obtained via the Darboux transformation of the complex KdV equation. Furthermore, the preferences for the nonsingular solutions are discussed.

Key words: Darboux Transformation; Positons; Negatons; Complexitons.

1. Introduction

It is well known that the Korteweg – de Vries (KdV) equation is one of the most important nonlinear evolution equations which can be used to describe weak shallow-water waves, collisionless plasma magneto-hydrodynamic waves, and long waves in anharmonic crystals. In the new classification based on the property of the spectral parameter, positon and negaton solutions are associated with real eigenvalues [1, 2] and complexiton solutions with complex eigenvalues [3].

However, both the positon and the complexiton solutions for the real KdV equation are singular. Recently, various general coupled KdV systems are derived from some different physical problems such as the two-wave modes in a shallow stratified liquid [4], two-layer models of atmospheric dynamical systems [5], and two-component Bose-Einstein condensates [6]. One of the common special cases of the above three different physical models reads

$$\begin{aligned} &U_t + 6VV_x - 6UU_x + U_{xxx} = 0, \\ &V_t - 6UV_x - 6VU_x + V_{xxx} = 0. \end{aligned} \tag{1}$$

Some special types of nonsingular positon, negaton and complexiton solutions of the coupled KdV equation (1) are given by Hu, Tong and Lou in [7].

However, there are many important problems, which should be clarified. One of the most important one is the singular or nonsingular problem. In [7], only the analytical positons, negatons and complexitons under some special parameter choices have been found for the coupled KdV equation (1). Usually, all these kinds of excitations can be singular. Then the following questions arise: Are there any singular positons, negatons and complexitons found for the coupled KdV equation (1) too? How can one find these kinds of singular solutions if they exist and what are the conditions for these types of exact solutions to be nonsingular?

In this paper, we will restudy the exact solutions of the coupled KdV equation (1) via the Darboux transformation (DT) of the complex KdV:

$$u_t + 6uu_x + u_{xxx} = 0, \tag{2}$$

because the coupled KdV system (1) can be derived simply from the real and imaginary parts (2) with

$$u = -U + \mathrm{i}V, \quad \mathrm{i} \equiv \sqrt{-1}. \tag{3}$$

Section 2 of this paper is devoted to find both the singular and the nonsingular positons, negatons and complexitons with the help of the first step DT. It is found that a single complexiton obtained from the first step DT is always singular except that the spectral parameter is real, which is related to the positon and/or negaton case. Under some special parameter choices, both the positons and negatons can be nonsingular. In Section 3, we discuss the exact solutions of the coupled KdV equation (1) via the second step DT. It is found that there are rich interaction structures for the two-complexiton interaction solutions obtained from the second step DT. Under some spectral parameter choices, the analytical positon-positon, positon-negaton and negaton-negaton solutions can be found. Two conjugate complexitons may form a special *analytical* complexiton, a *breather*. However, we have not

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yet found other types of analytical interaction solutions such as complexiton-negaton interactions. The last section contains a short summary and discussion.

2. Exact Solutions from the First Step Darboux Transformation

It is clear that the Lax pair for the complex KdV equation (2) has the form

$$\phi_{xx} = -u\phi - \lambda\,\phi\,, \quad \phi_t = -4\,\phi_{xxx} - 6\,u\phi_x - 3\,u_x\phi\,, \quad (4)$$

and the corresponding first step DT is

$$u' = u + 2(\ln f)_{xx}\,, \quad \phi' = \phi_x - \frac{f_x}{f}\phi\,, \quad (5)$$

where f is a wave function solution of the Lax pair (4) with $\lambda = \lambda_0$.

It is known that the complexiton solutions are linked with the complex spectral parameters. Without lost of generality, let the spectral parameter $\lambda = \lambda_0 = \lambda_1 + \mathrm{i}\lambda_2 \equiv (\alpha^2 - \beta^2) + \mathrm{i}(2\alpha\beta)$ and the seed solution $u = 0$, one can directly obtain the corresponding spectral function of the Lax pair (4). The final result reads

$$f = \cosh\left[(\mathrm{i}\alpha - \beta)x - 4(\mathrm{i}\alpha - \beta)^3 t + \delta_1 + \mathrm{i}\delta_2\right], \quad (6)$$

where the parameters $\{\alpha, \beta, \delta_1, \delta_2\}$ are all real.

Substituting (6) into (5) yields a single complexiton solution [3] for the complex KdV equation (2):

$$u = \frac{2(\mathrm{i}\alpha - \beta)^2}{\cosh^2\left[(\mathrm{i}\alpha - \beta)x - 4(\mathrm{i}\alpha - \beta)^3 t + \delta_1 + \mathrm{i}\delta_2\right]}. \quad (7)$$

In general, the complexiton (7) has a singularity located at

$$x = -\left[-6\alpha\delta_1\beta^2 + 2\alpha^3\delta_1 - 2\beta^3\delta_2 + \beta^3\pi k + 6\alpha^2\delta_2\beta - 3\alpha^2\pi k\beta\right]\left[4\alpha\beta(\beta^2 + \alpha^2)\right]^{-1},$$
$$t = \frac{2\alpha\delta_1 + 2\delta_2\beta - \pi k\beta}{16\alpha\beta(\beta^2 + \alpha^2)} \quad (k = 2n + 1,\, n \in Z). \quad (8)$$

It is clear that solution (7) is always singular when $\alpha \neq 0$ and $\beta \neq 0$. Therefore, the nonsingular conditions (if exists) can only be $\alpha = 0$ or $\beta = 0$, namely the spectral parameter is associated with the real eigenvalue.

In the remaining part of this paper, we mainly focus on the nonsingular solutions such as the nonsingular negatons, positons and complexitons.

2.1. Negaton Solution ($\lambda = -\beta^2 < 0$, $\alpha = 0$)

In this special case, the solution (7) becomes a single negaton solution:

$$u = \frac{2\beta^2}{\cosh^2(-x\beta + 4t\beta^3 + \delta_1 + \mathrm{i}\delta_2)}. \quad (9)$$

The negaton solution (9) is singular only for $\delta_2 = (k + 1/2)\pi$, $k \in Z$.

Separating the real and imaginary parts of (9), and using (3), we obtain the negaton solutions for coupled KdV equation:

$$U = \frac{-\beta^2\left[\cos(2\delta_2)\cosh(2\beta x - 8\beta^3 t - 2\delta_1) + 1\right]}{\left[\cosh^2(\beta x - 4\beta^3 t - \delta_1) - \sin^2\delta_2\right]^2},$$
$$V = \frac{\beta^2\sin(2\delta_2)\sinh(2x\beta - 8t\beta^3 - 2\delta_1)}{\left[\cosh^2(\beta x - 4\beta^3 t - \delta_1) - \sin^2\delta_2\right]^2}. \quad (10)$$

It is interesting that the analytical negaton solution (10) for U has two different kinds of characteristic structures. The first kind of solution (for simplicity we

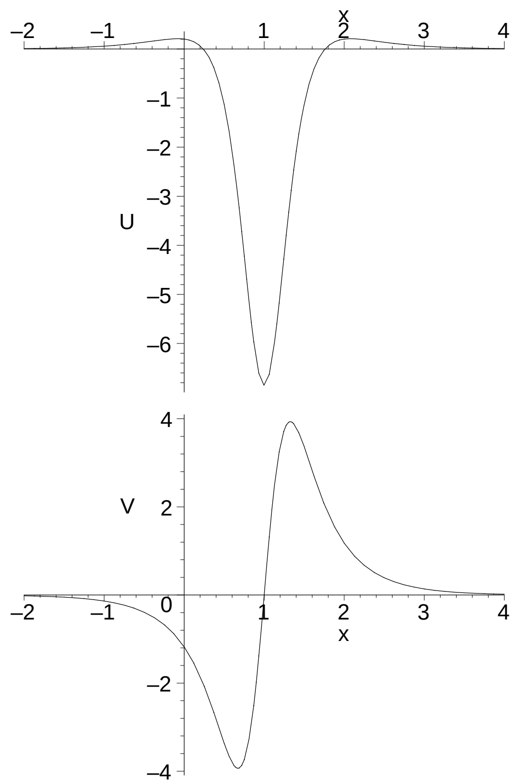

Fig. 1. Negaton solution (N_1-type) of the coupled KdV system expressed by (10) under the parameter choice $\beta = 1.0$, $\delta_1 = 1.0$, $\delta_2 = 1.0$ at the time $t = 0$.

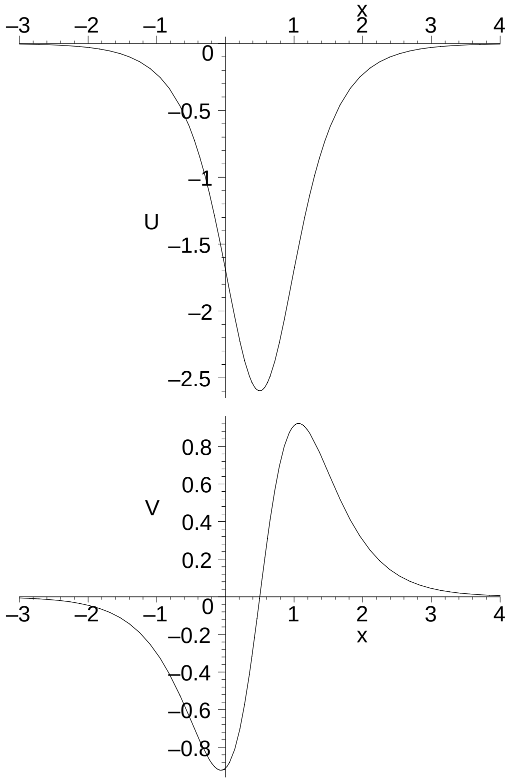

Fig. 2. Negaton solution (N_2-type) of the coupled KdV system expressed by (10) under the parameter choice $\beta = 1.0$, $\delta_1 = 0.5$, $\delta_2 = 0.5$ at the time $t = 0$.

later call it N_2) possesses a similar structure to the soliton solution of the real KdV equation $U < 0$ with one single extremum. The second kind of negaton solution (for simplicity we call it N_1) possesses a completely different kind of structure from that of the real KdV equation: there are three extrema and $U < 0$ is valid only in the neighbourhood of the center of the soliton if the same boundary conditions $U(\pm\infty) = 0$ are satisfied. To demonstrate this point, we write down the positions of the extrema for the field U:

$$x_l = \frac{1}{2\beta}\left[8t\beta^3 + 2\delta_1 - \ln\left(2 + 8\xi^2 - 8\xi^4 - 4\sqrt{4\xi^8 - 8\xi^6 + \xi^4 + 3\xi^2}\right) + \ln(2 - 4\xi^2)\right],$$

$$x_r = \frac{1}{2\beta}\left[8t\beta^3 + 2\delta_1 - \ln\left(2 + 8\xi^2 - 8\xi^4 + 4\sqrt{4\xi^8 - 8\xi^6 + \xi^4 + 3\xi^2}\right) + \ln(2 - 4\xi^2)\right],$$

$$x_m = \frac{4t\beta^3 + \delta_1}{\beta}, \quad \xi = \cos(\delta_2). \tag{11}$$

Substituting (11) into (10), one can easy get the extrema for the field U:

$$U(x_l) = U(x_r) = \frac{\beta^2}{\tan^2(2\delta_2)}, \quad U(x_m) = \frac{-2\beta^2}{\cos^2(\delta_2)}.$$

From (11) we find that U expressed by (10) possesses three extrema in the range $-\frac{1}{\sqrt{2}} < \xi < \frac{1}{\sqrt{2}}$, namely $(k+1/4)\pi < \delta_2 < (k+3/4)\pi$, $k \in Z$. However, if ξ is determined by $(k-1/4)\pi < \delta_2 < (k+1/4)\pi$, $k \in Z$, U possesses only one extremum, x_m. The typical structure of these two kinds of negaton solution are shown in Figs. 1 and 2, respectively.

2.2. Positon Solution ($\lambda = \alpha^2 > 0, \beta = 0$)

In the same way, the solution (7) becomes a single positon solution:

$$u = \frac{-2\alpha^2}{\cosh^2(\mathrm{i}x\alpha + 4\mathrm{i}\alpha^3 t + \delta_1 + \mathrm{i}\delta_2)}, \tag{12}$$

which is singular only for $\delta_1 = 0$.

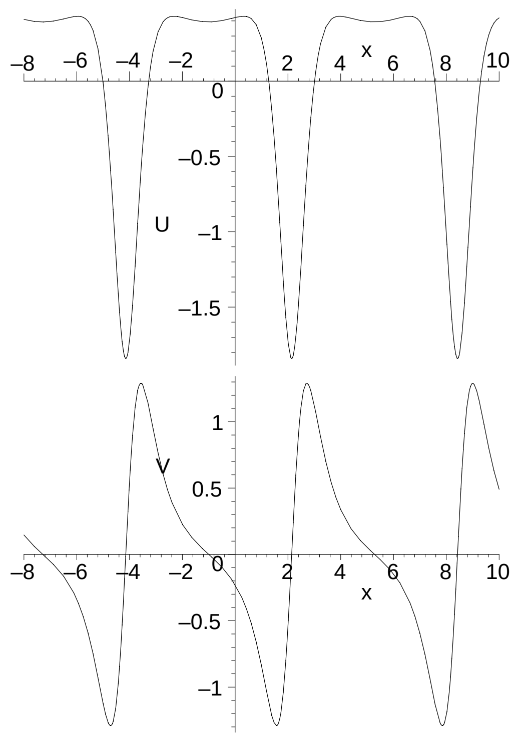

Fig. 3. Positon solution (P_1-type) of the coupled KdV system expressed by (13) under the parameter choice $\alpha = 0.5$, $\delta_1 = -0.5$, $\delta_2 = 0.5$ at the time $t = 0$.

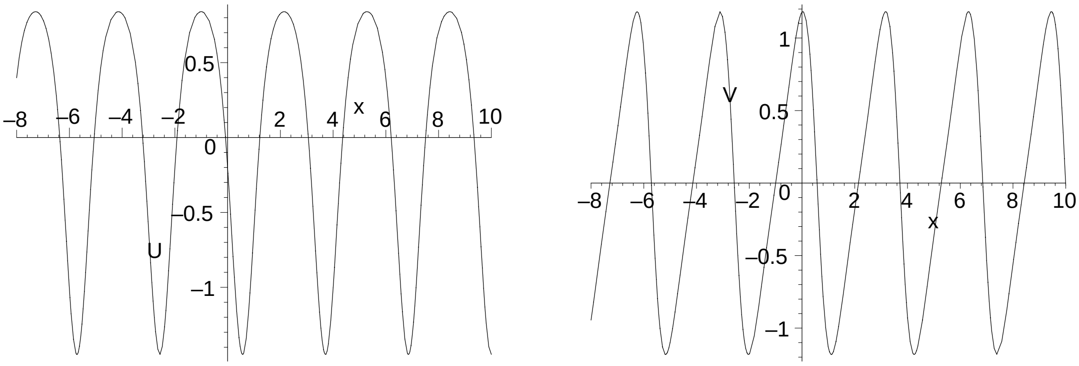

Fig. 4. Positon solution (P_2-type) of the coupled KdV system expressed by (13) under the parameter choice $\alpha = 1.0$, $\delta_1 = 1.0$, $\delta_2 = 1.0$ at the time $t = 0$.

Writing down the real and imaginary parts of (12), we obtain the positon solutions for the coupled KdV equation:

$$U = \frac{\alpha^2\left[\cosh(2\delta_1)\cos(2\alpha x+8\alpha^3 t+2\delta_2)+1\right]}{\left[\cos^2(\alpha x+4\alpha^3 t+\delta_2)+\sinh^2\delta_1\right]^2}, \quad V = \frac{\alpha^2\sin(2\alpha x+8\alpha^3 t+2\delta_2)\sinh(2\delta_1)}{\left[\cos^2(\alpha x+4\alpha^3 t+\delta_2)+\sinh^2\delta_1\right]^2}. \tag{13}$$

As for the negaton case, there are two kinds of characteristic structures for the positon solution (13). The first type of positons (P_1) possesses four extrema in one period and the second type of positon solution (P_2) possesses only two extrema. To see this more clearly, we write down the location of the four extrema for the field U at the range $0 \le \alpha x + 4\alpha^3 t + \delta_2 < \pi$. The result reads

$$\begin{aligned}
x_1 &= \frac{1}{2\alpha}\left[\arctan\left(\frac{(\xi^4-1)\sqrt{-1-\xi^8+14\xi^4}}{\xi^2(\xi^4+1)}, \frac{\xi^8-6\xi^4+1}{\xi^2(\xi^4+1)}\right) - 8t\alpha^3 - 2\delta_2\right],\\
x_2 &= \frac{1}{2\alpha}\left[\arctan\left(\frac{-(\xi^4-1)\sqrt{-1-\xi^8+14\xi^4}}{\xi^2(\xi^4+1)}, \frac{\xi^8-6\xi^4+1}{\xi^2(\xi^4+1)}\right) - 8t\alpha^3 - 2\delta_2\right],\\
x_3 &= -\frac{4t\alpha^3+\delta_2}{\alpha}, \quad x_4 = -\frac{4t\alpha^3+\delta_2}{\alpha} + \frac{\pi}{2\alpha}, \qquad \xi = \exp(\delta_1).
\end{aligned} \tag{14}$$

One can easy get the corresponding extremum of the field U:

$$U(x_1) = U(x_2) = \frac{\alpha^2}{\tanh^2(2\delta_1)},$$

$$U(x_3) = \frac{2\alpha^2}{\cosh^2(\delta_1)}, \quad U(x_4) = \frac{-2\alpha^2}{\sinh^2(\delta_1)}.$$

It is clear that in every period, if $\frac{\sqrt{6}-\sqrt{2}}{2} < \xi < \frac{\sqrt{6}+\sqrt{2}}{2}$, namely $\ln(\frac{\sqrt{6}-\sqrt{2}}{2}) < \delta_1 < \ln(\frac{\sqrt{6}+\sqrt{2}}{2})$, there are four extrema at $x = x_1, x_2, x_3$, and x_4. When ξ fall into the intervals $\delta_1 < \ln(\frac{\sqrt{6}-\sqrt{2}}{2})$ or $\delta_1 > \ln(\frac{\sqrt{6}+\sqrt{2}}{2})$, there are only two extrema located at $x = x_3$ and x_4. Figure 3 is a typical plot of a P_1-type positon solution, while Fig. 4 is a characteristic structure of a P_2-type one.

3. Interaction Solutions from the Second Step Darboux Transformation

In the previous section, we have obtained the single nonsingular negaton and positon solutions to the coupled KdV equation via the first step DT of the complex KdV equation by taking the seed solution $u = 0$. The complexitons obtained from the first step DT are all singular. In order to get analytic complexitons and interaction solutions among negatons, positons and complexitons we have to use multiple step DTs. In this section, we discuss the interaction solutions via the second step DT.

In the same way, taking the seed solution $u = 0$, then using the spectral functions f_1 and f_2 with different spectral parameters given by (6), and let

$$\psi = f_{1x} f_2 - f_{2x} f_1, \tag{15}$$

the solution of the complex KdV equation obtained from the second step DT reads

$$u = 2(\ln \psi)_{xx}. \tag{16}$$

If F and G are the real and imaginary parts of ψ given by (15), respectively, then the corresponding solution of the coupled KdV equation has the form

$$U = -[\ln(F^2 + G^2)]_{xx}, \quad V = 2\left[\arctan\frac{G}{F}\right]_{xx}. \tag{17}$$

3.1. Negaton Interaction Solution

Taking two spectral functions as follows:

$$\begin{aligned} f_1 &= \cosh(-\beta_1 x + 4\beta_1^3 t + \delta_{11} + \mathrm{i}\delta_{21}), \\ f_2 &= \cosh(-\beta_2 x + 4\beta_2^3 t + \delta_{12} + \mathrm{i}\delta_{22}), \end{aligned} \tag{18}$$

the Wronskian of f_1 and f_2, i. e., ψ becomes

$$\begin{aligned} \psi = &-\beta_2 \sinh(\beta_2 x - 4\beta_2^3 t - \delta_{12} - \mathrm{i}\delta_{22}) \\ &\cdot \cosh(\beta_1 x - 4\beta_1^3 t - \delta_{11} - \mathrm{i}\delta_{21}) \\ &+ \beta_1 \cosh(\beta_2 x - 4\beta_2^3 t - \delta_1 2 - \mathrm{i}\delta_{22}) \\ &\cdot \sinh(\beta_1 x - 4\beta_1^3 t - \delta_{11} - \mathrm{i}\delta_{21}). \end{aligned} \tag{19}$$

The real and imaginary parts of (19), F and G, can be written as

$$\begin{aligned} F = &\frac{\beta_1+\beta_2}{2}\cos(\delta_{21}-\delta_{22})\sinh(A_1 - A_2) \\ &+ \frac{\beta_1-\beta_2}{2}\cos(\delta_{21}+\delta_{22})\sinh(A_1 + A_2), \\ G = &\frac{\beta_2-\beta_1}{2}\sin(\delta_{21}+\delta_{22})\cosh(A_1 + A_2) \\ &- \frac{\beta_1+\beta_2}{2}\sin(\delta_{21}-\delta_{22})\cosh(A_1 - A_2), \end{aligned} \tag{20}$$

where $A_1 = \beta_1 x - 4\beta_1^3 t - \delta_{11}$ and $A_2 = \beta_2 x - 4\beta_2^3 t - \delta_{12}$.

So the negaton interaction solution of the complex KdV equation can be expressed by (16) and (19), and the solution of the coupled KdV equation can be obtained by substituting (20) into (17).

One finds that the solution (17) with (20) is nonsingular, if the parameters satisfy

$$\begin{aligned} (\beta_2^2 - \beta_1^2)\big[&\beta_1^2\cos^2(\delta_{22}) - \beta_1\beta_2\cot(2\delta_{21})\sin(2\delta_{22}) \\ &- \beta_2^2\sin^2(\delta_{22})\big] < 0 \end{aligned}$$

or

$$\begin{aligned} (\beta_2^2 - \beta_1^2)\big[&\beta_1^2\sin^2(\delta_{21}) - \beta_1\beta_2\cot(2\delta_{22})\sin(2\delta_{21}) \\ &- \beta_2^2\cos^2(\delta_{21})\big] < 0. \end{aligned}$$

Unlike the single negaton solution case, one can not directly write down the explicit form of the positions of the extrema for the field U. To demonstrate the different kinds of characteristic structures, firstly, we give the approximate form of the field U for large times. The result reads

$$\begin{aligned} U\mid_{t\to\pm\infty} \approx\ &W_1[\mathrm{sign}(\pm\beta_1\beta_2^2 \mp \beta_1^3)] \\ &+ W_2[\mathrm{sign}(\pm\beta_1^2\beta_2 \mp \beta_2^3)], \end{aligned} \tag{21}$$

where

$$W_1[\pm] = U|_{A_1\to\pm\infty} = U|_{\cosh(A_1)=\pm\sinh(A_1)},$$
$$W_2[\pm] = U|_{A_2\to\pm\infty} = U|_{\cosh(A_2)=\pm\sinh(A_2)}.$$

It is found that $W_1[\pm]$ and $W_2[\pm]$ is independent with respect to A_1 and A_2, respectively. Moreover, both of them possess the characteristic structures similar to the single negaton solution. To demonstrate this point, we write down the positions of the extrema with respect to $W_1[\pm]$:

$$\begin{aligned} A_2^r[\pm] &= \frac{1}{2}\ln\left[\frac{(\beta_1\pm\beta_2)(1+6\xi_2^2+\xi_2^4+2\sqrt{3\xi_2^6+10\xi_2^4+3\xi_2^2})}{(\beta_1\mp\beta_2)(\xi_2^4-1)}\right], \\ A_2^l[\pm] &= \frac{1}{2}\ln\left[\frac{(\beta_1\pm\beta_2)(1+6\xi_2^2+\xi_2^4-2\sqrt{3\xi_2^6+10\xi_2^4+3\xi_2^2})}{(\beta_1\mp\beta_2)(\xi_2^4-1)}\right], \\ A_2^m[\pm] &= \frac{1}{2}\ln\left|\frac{\beta_1\pm\beta_2}{\beta_1\mp\beta_2}\right|, \quad \xi_2 = \tan(\delta_{22}), \end{aligned} \tag{22}$$

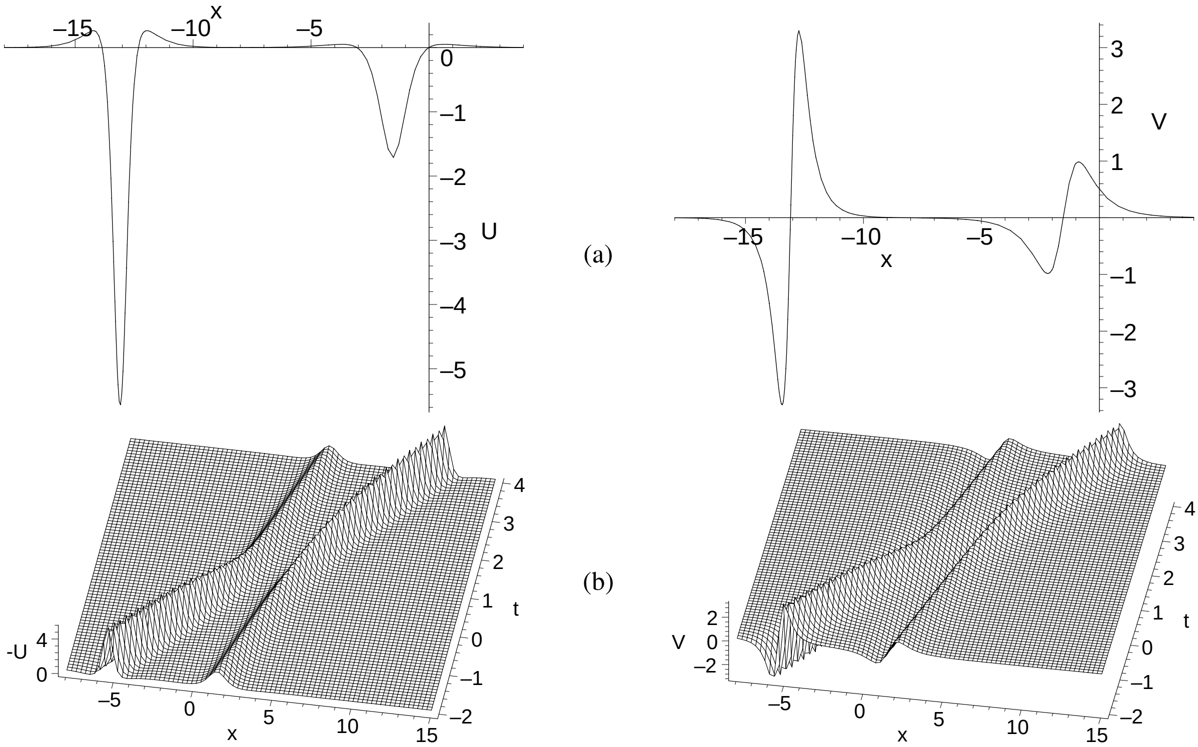

Fig. 5. Negaton interaction solution (N_1-N_1-type) with the parameter choice $\beta_1 = 0.5$, $\beta_2 = 0.8$, $\delta_{11} = 1.0$, $\delta_{21} = 1.0$, $\delta_{12} = 0.5$, $\delta_{22} = -0.5$; (a) characteristic waveform at $t = -5$; (b) spatio-temporal evolution plot.

and the positions of the extrema with respect to $W_2[\pm]$:

$$\begin{aligned}
A_1^r[\pm] &= \frac{1}{2}\ln\left[\frac{(\beta_1 \pm \beta_2)(1 + 6\xi_1^2 + \xi_1^4 + 2\sqrt{3\xi_1^6 + 10\xi_1^4 + 3\xi_1^2})}{(\beta_1 \mp \beta_2)(1 - \xi_1^4)}\right], \\
A_1^l[\pm] &= \frac{1}{2}\ln\left[\frac{(\beta_1 \pm \beta_2)(1 + 6\xi_1^2 + \xi_1^4 - 2\sqrt{3\xi_1^6 + 10\xi_1^4 + 3\xi_1^2})}{(\beta_1 \mp \beta_2)(1 - \xi_1^4)}\right], \\
A_1^m[\pm] &= \frac{1}{2}\ln\left|\frac{\beta_1 \pm \beta_2}{\beta_1 \mp \beta_2}\right|, \quad \xi_1 = \tan(\delta_{21}).
\end{aligned} \tag{23}$$

Then the extremum with respect to W_1 and W_2 reads

$$W_1(A_2^l) = W_1(A_2^r) = \frac{\beta_2^2}{\tan^2(2\delta_{22})},$$

$$W_1(A_2^m) = \frac{-4\beta_2^2}{1 + \mathrm{sign}(\beta_1^2 - \beta_2^2)\cos(2\delta_{22})},$$

$$W_2(A_1^l) = W_2(A_1^r) = \frac{\beta_1^2}{\tan^2(2\delta_{21})},$$

$$W_2(A_1^m) = \frac{-4\beta_1^2}{1 + \mathrm{sign}(\beta_2^2 - \beta_1^2)\cos(2\delta_{21})}.$$

It is clear that W_1 possesses three extrema with parameter choice if $(\beta_1^2 - \beta_2^2)(\xi_2^4 - 1) > 0$, otherwise only one extremum if $(\beta_1^2 - \beta_2^2)(\xi_2^4 - 1) < 0$. The similar result can be easily obtained with respect to W_2. So, the negaton interaction solution for U has three kinds of characteristic structures as shown in Figs. 5, 6, and 7.

3.2. Negaton-Positon Interaction Solution

Taking the two spectral functions as

$$\begin{aligned}
f_1 &= \cosh(-\beta x + 4\beta^3 t + \delta_{11} + \mathrm{i}\delta_{21}), \\
f_2 &= \cosh(\mathrm{i}\alpha x + 4\mathrm{i}\alpha^3 t + \delta_{12} + \mathrm{i}\delta_{22}),
\end{aligned} \tag{24}$$

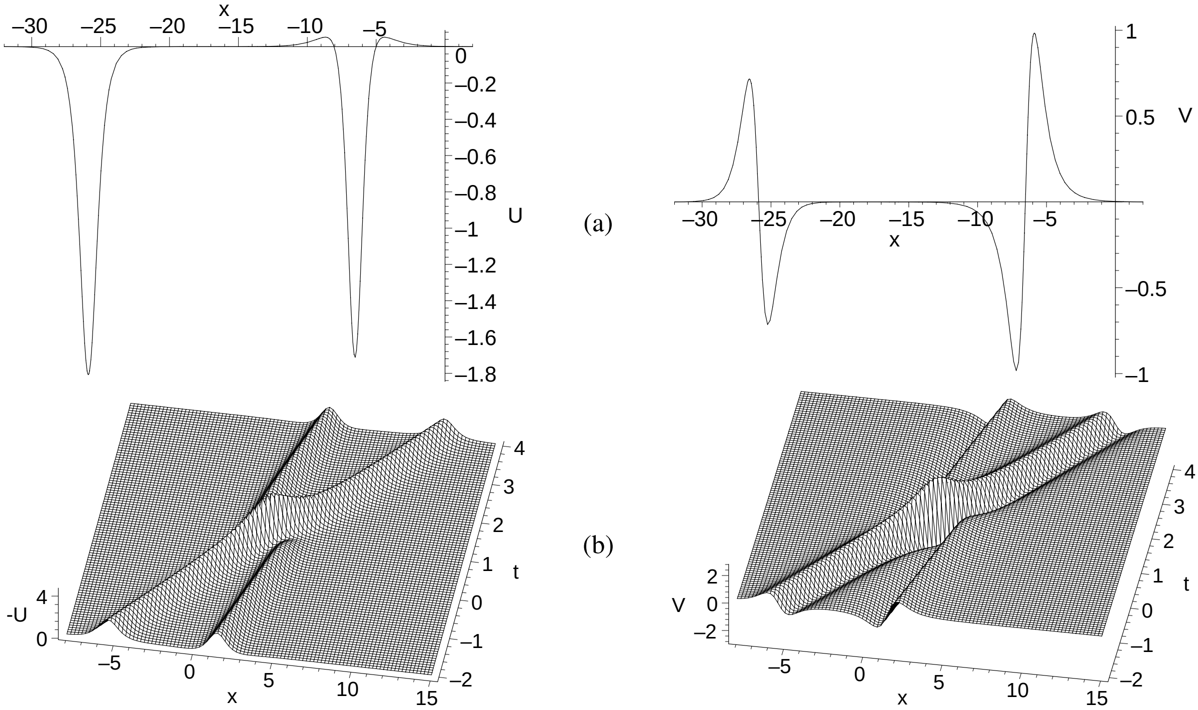

Fig. 6. Negaton interaction solution (N_1-N_2-type) with the parameter choice $\beta_1 = 0.5$, $\beta_2 = 0.8$, $\delta_{11} = 1.0$, $\delta_{21} = 1.0$, $\delta_{12} = 0.5$, $\delta_{22} = 1.0$; (a) characteristic waveform at $t = -10$; (b) spatio-temporal evolution plot.

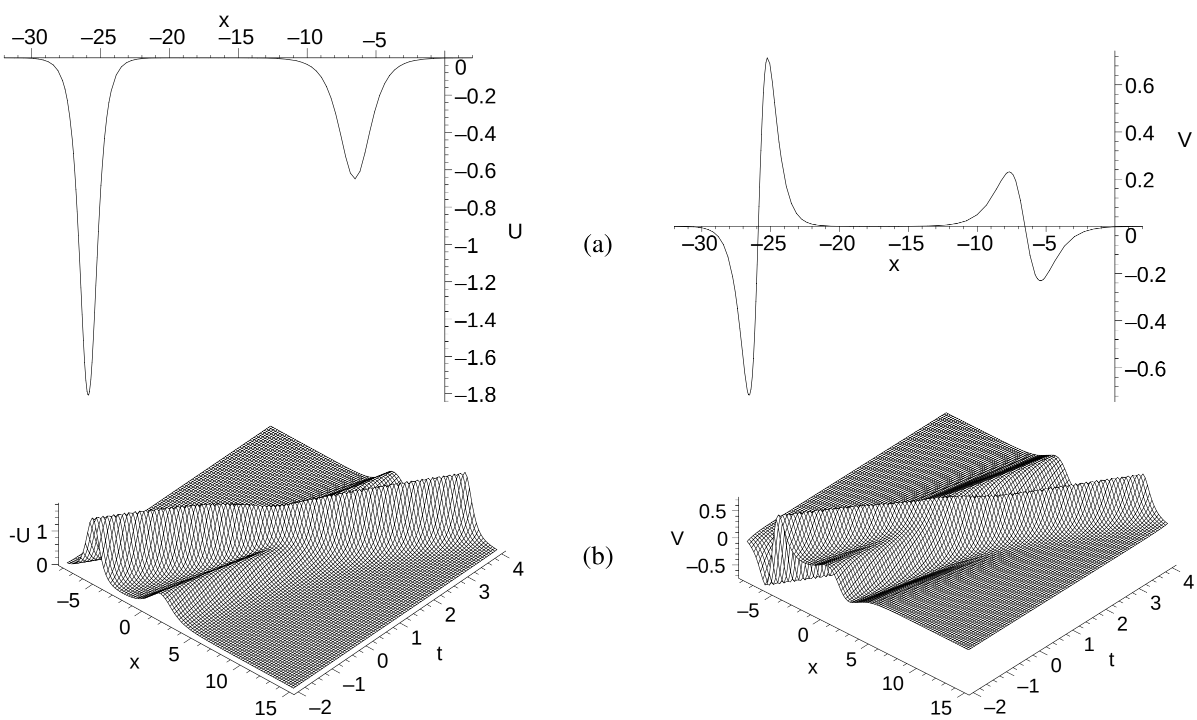

Fig. 7. Negaton interaction solution (N_2-N_2-type) with the parameter choice $\beta_1 = 0.5$, $\beta_2 = 0.8$, $\delta_{11} = 1.0$, $\delta_{21} = -0.5$, $\delta_{12} = 0.5$, $\delta_{22} = -1.0$; (a) characteristic waveform at $t = -10$; (b) spatio-temporal evolution plot.

then

$$\begin{aligned}\psi = &-\beta\sinh(-\beta x + 4\beta^3 t + \delta_{11} + \mathrm{i}\delta_{21}) \\ &\cdot\cos(\alpha x + 4\alpha^3 t - \mathrm{i}\delta_{12} + \delta_{22}) \\ &+ \alpha\sin(\alpha x + 4\alpha^3 t - \mathrm{i}\delta_{12} + \delta_{22}) \\ &\cdot\cosh(-\beta x + 4\beta^3 t + \delta_{11} + \mathrm{i}\delta_{21})\end{aligned} \quad (25)$$

and

$$\begin{aligned}F = &+\frac{\alpha}{2}\{\sin(\delta_{21}+A_2)\cosh(A_1+\delta_{12}) \\ &+ \sin(-\delta_{21}+A_2)\cosh(A_1-\delta_{12})\} \\ &-\frac{\beta}{2}\{\cos(-\delta_{21}+A_2)\sinh(A_1-\delta_{12}) \\ &+ \cos(\delta_{21}+A_2)\sinh(A_1+\delta_{12})\}, \\ G = &+\frac{\alpha}{2}\{\cos(-\delta_{21}+A_2)\sinh(A_1-\delta_{12}) \\ &- \cos(\delta_{21}+A_2)\sinh(A_1+\delta_{12})\} \\ &-\frac{\beta}{2}\{\sin(\delta_{21}+A_2)\cosh(A_1+\delta_{12}) \\ &- \sin(-\delta_{21}+A_2)\cosh(A_1-\delta_{12})\},\end{aligned} \quad (26)$$

where $A_1 = -\beta x + 4\beta^3 t + \delta_{11}$, $A_2 = \alpha x + 4\alpha^3 t + \delta_{22}$.

So the negaton-positon interaction solution to the complex KdV equation can be expressed by (16) and (25), and the solution to the coupled KdV equation can be obtained by substituting (26) into (17).

The nonsingularity condition now reads

$$\beta^2\sin^2(\delta_{21}) + \alpha\beta\coth(2\delta_{12})\sin(2\delta_{21}) + \alpha^2\cos^2(\delta_{21}) > 0$$

or

$$\alpha^2\cosh^2(\delta_{12}) - \alpha\beta\cot(2\delta_{21})\sinh(2\delta_{12}) - \beta^2\sinh^2(\delta_{12}) < 0$$

or

$$\alpha^2\cosh^2(\delta_{12}) - \alpha\beta\cot(2\delta_{21})\sinh(2\delta_{12}) - \beta^2\sinh^2(\delta_{12}) > 1.$$

To demonstrate the different kinds of characteristic structures of the negaton-positon interaction solution, firstly, we take the approximate form of the field U for the large spatial variable x. The result reads

$$U\,|_{x\to\pm\infty} \approx W[\mathrm{sign}(\mp\beta)], \quad (27)$$

where

$$W[\pm] = U|_{A_1\to\pm\infty} = U|_{\cosh(A_1)=\pm\sinh(A_1)}.$$

It is found that $W[\pm]$ is independent with respect to A_1. Moreover, the characteristic structure of $W[\pm]$ is similar to the single positon solution. To demonstrate this point, we write down the positions of the extrema with respect to $W[\pm]$ which satisfy

$$\begin{aligned}&\cos(A_2^1) = \frac{\sqrt{\alpha^2\xi^2(3-2\xi^2)}+\sqrt{\beta^2(\xi^2-1)(2\xi^2+1)}}{\sqrt{(\alpha^2+\beta^2)(2\xi^2-1)}}, \\ &\cos(A_2^2) = \frac{\sqrt{\beta^2\xi^2(3-2\xi^2)}-\sqrt{\alpha^2(\xi^2-1)(2\xi^2+1)}}{\sqrt{(\alpha^2+\beta^2)(2\xi^2-1)}}, \\ &\cos(A_2^3) = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}, \quad \cos(A_2^4) = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}, \\ &\xi = \cosh(\delta_{12}).\end{aligned} \quad (28)$$

Then we have the extremum with respect to W:

$$\begin{aligned}&W(A_2^1) = W(A_2^2) = \frac{\alpha^2}{\tanh^2(2\delta_{12})}, \\ &W(A_2^3) = \frac{2\alpha^2}{\cosh^2(\delta_{12})}, \quad W(A_2^4) = \frac{-2\alpha^2}{\sinh^2(\delta_{12})}.\end{aligned}$$

It is clear that W possesses four extrema with parameter choice if $\xi^2 < \frac{3}{2}$, otherwise it possesses only two extrema. Unfortunately, we find it is not an easy job to reduce the negaton-positon interaction solution to a simple function only with respect to A_1 by the approximate method. One of the possible ways is just taking A_2 as constant. For simplicity let $A_2 = 0$, then one can easily find that the extremum property is similar to a single negaton, namely for some parameter choice the solution possesses three extrema and for others only one. However, the result is too lengthy to be written down here. To sum up, the negaton-positon interaction solution has four kinds of characteristic structures as shown in Figures 8 – 11.

3.3. Positon Interaction Solution

Taking the two spectral functions

$$\begin{aligned}&f_1 = \cosh\left(\mathrm{i}\alpha_1 x + 4\mathrm{i}\alpha_1^3 t + \delta_{11} + \mathrm{i}\delta_{21}\right), \\ &f_2 = \cosh\left(\mathrm{i}\alpha_2 x + 4\mathrm{i}\alpha_2^3 t + \delta_{12} + \mathrm{i}\delta_{22}\right),\end{aligned} \quad (29)$$

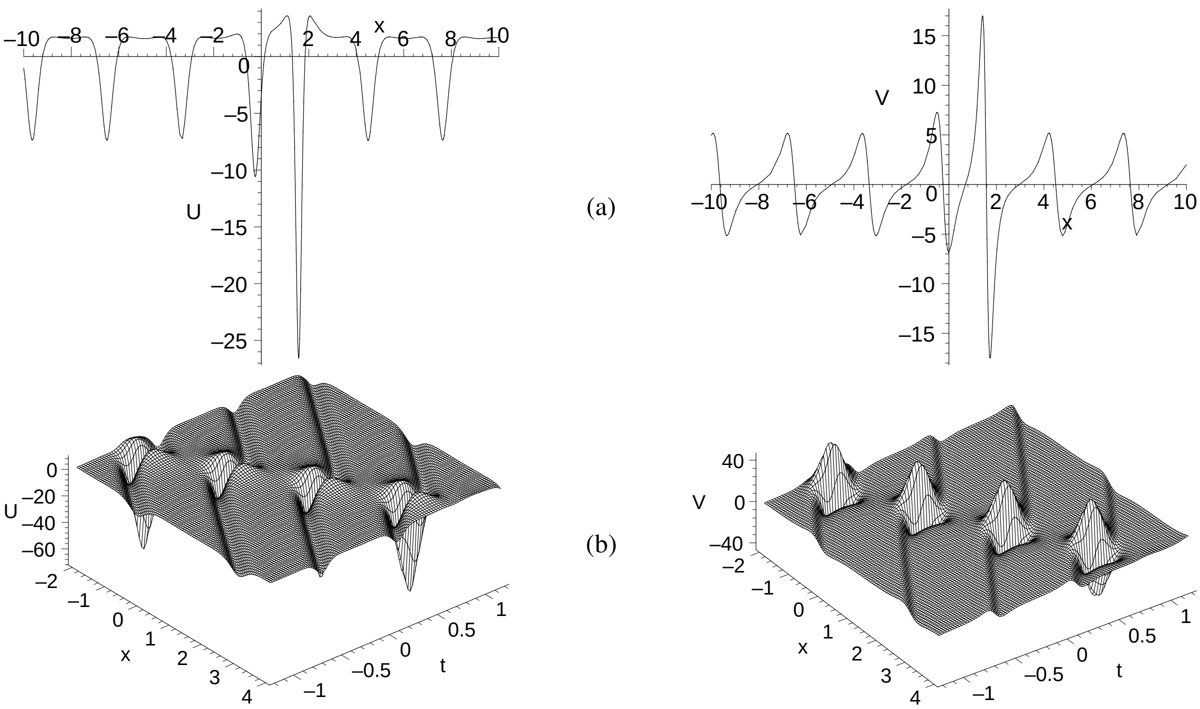

Fig. 8. Negaton-positon interaction solution (N_1-P_1-type) with the parameter choice $\alpha = 1.0$, $\beta = 1.0$, $\delta_{11} = 1.0$, $\delta_{21} = 3.0$, $\delta_{12} = 0.5$, $\delta_{22} = 1.0$; (a) characteristic waveform at $t = 0$; (b) spatio-temporal evolution plot.

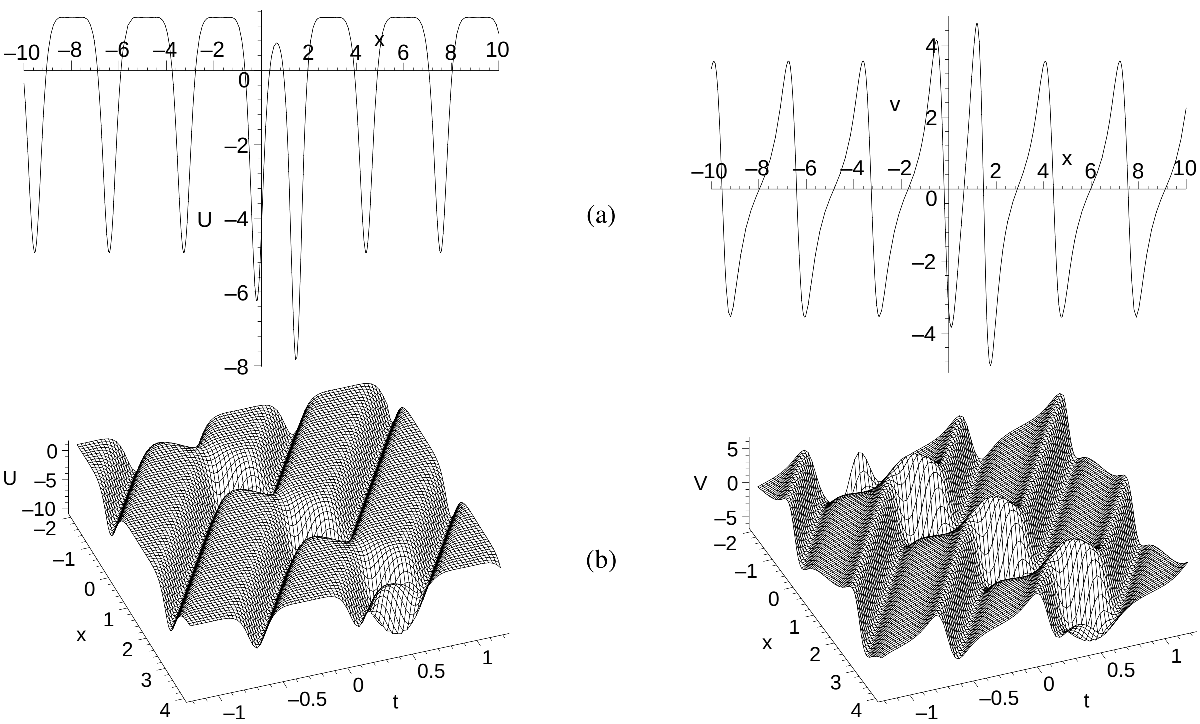

Fig. 9. Negaton-positon interaction solution (N_2-P_1-type) with the parameter choice $\alpha = 1.0$, $\beta = 1.2$, $\delta_{11} = 1.0$, $\delta_{21} = 0.4$, $\delta_{12} = 0.6$, $\delta_{22} = 1.0$; (a) characteristic waveform at $t = 0$; (b) spatio-temporal evolution plot.

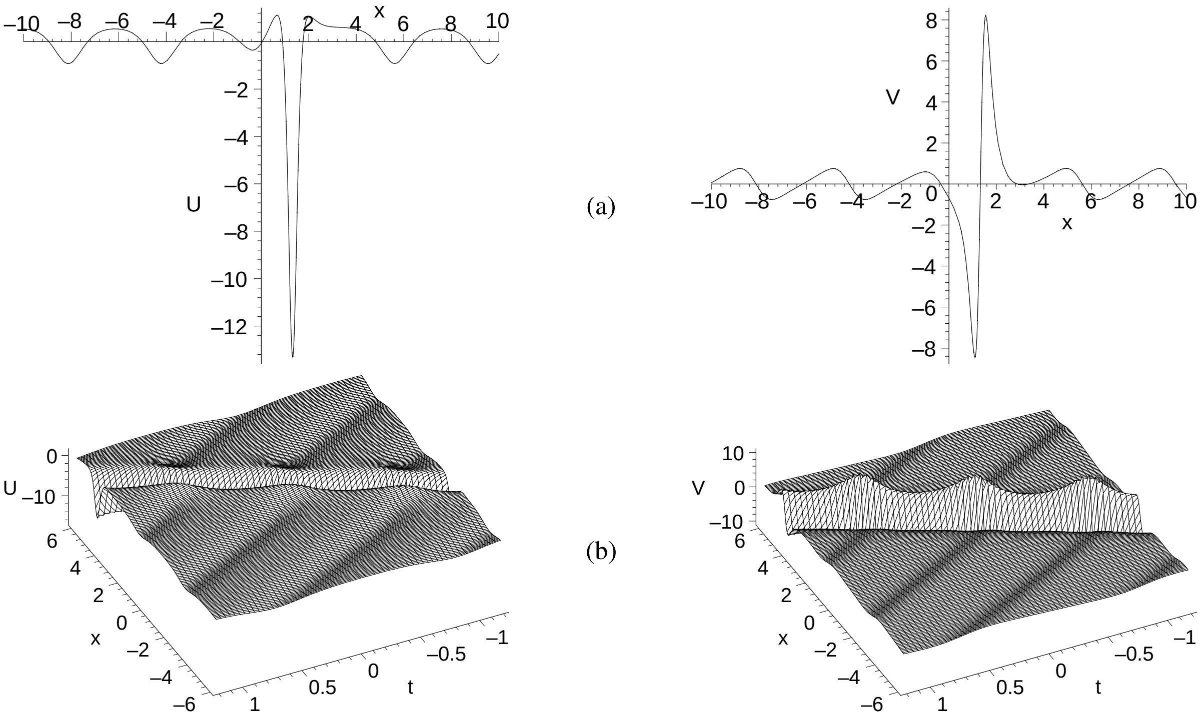

Fig. 10. Negaton-positon interaction solution (N_1-P_2-type) with the parameter choice $\alpha = 0.8$, $\beta = 0.8$, $\delta_{11} = 1.0$, $\delta_{21} = 2.0$, $\delta_{12} = 1.0$, $\delta_{22} = 1.0$; (a) characteristic waveform at $t = 0$; (b) spatio-temporal evolution plot.

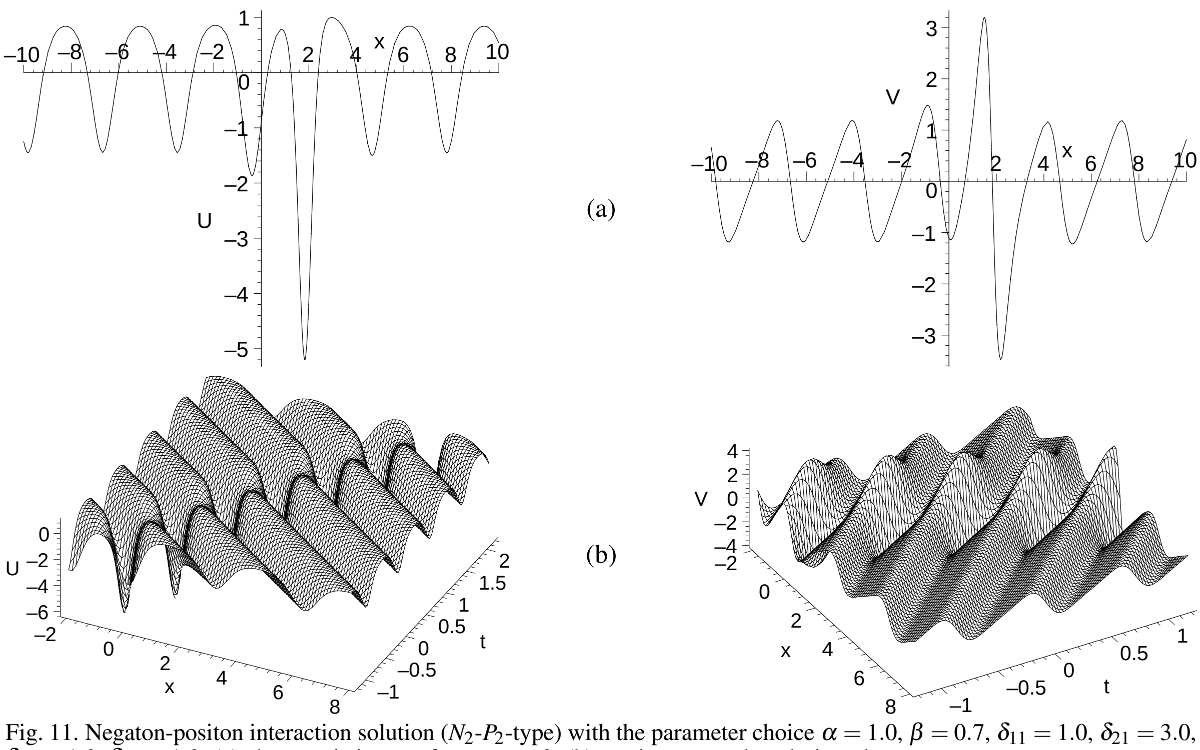

Fig. 11. Negaton-positon interaction solution (N_2-P_2-type) with the parameter choice $\alpha = 1.0$, $\beta = 0.7$, $\delta_{11} = 1.0$, $\delta_{21} = 3.0$, $\delta_{12} = 1.0$, $\delta_{22} = 1.0$; (a) characteristic waveform at $t = 0$; (b) spatio-temporal evolution plot.

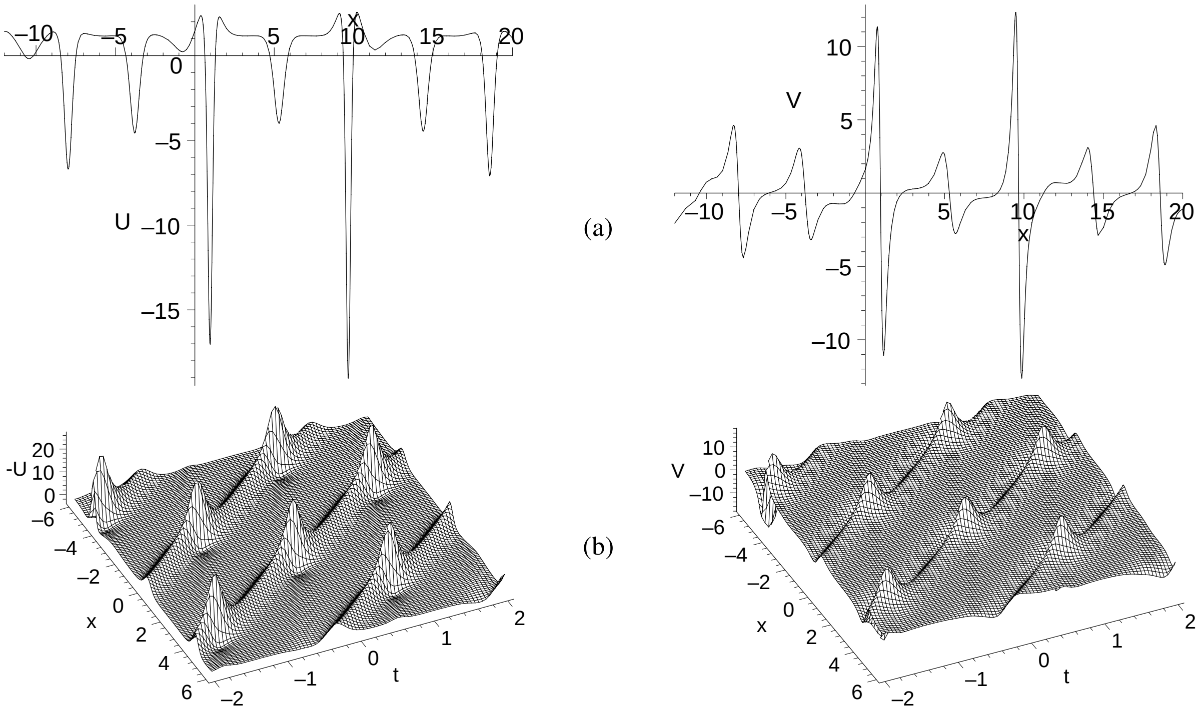

Fig. 12. Positon interaction solution (P_1-P_1-type) with the parameter choice $\alpha_1 = 1.0$, $\alpha_2 = 0.7$, $\delta_{11} = 1.0$, $\delta_{21} = 1.0$, $\delta_{12} = 0.5$, $\delta_{22} = 1.0$; (a) characteristic waveform at $t = 0$; (b) spatio-temporal evolution plot.

the Wronskian of f_1 and f_2 becomes

$$\begin{aligned}\psi = \mathrm{i}\{&\alpha_1 \sinh(\mathrm{i}\alpha_1 x + 4\mathrm{i}\alpha_1^3 t + \delta_{11} + \mathrm{i}\delta_{21})\\ &\cdot \cosh(\mathrm{i}\alpha_2 x + 4\mathrm{i}\alpha_2^3 t + \delta_{12} + \mathrm{i}\delta_{22})\\ &- \alpha_2 \sinh(\mathrm{i}\alpha_2 x + 4\mathrm{i}\alpha_2^3 t + \delta_{12} + \mathrm{i}\delta_{22})\\ &\cdot \cosh(\mathrm{i}\alpha_1 x + 4\mathrm{i}\alpha_1^3 t + \delta_{11} + \mathrm{i}\delta_{21})\}.\end{aligned} \tag{30}$$

The real and imaginary parts of (30), F and G, can be written as

$$\begin{aligned}F &= \frac{\alpha_2 - \alpha_1}{2} \sin(B_1 + B_2)\cosh(\delta_{11} + \delta_{12})\\ &\quad - \frac{\alpha_1 + \alpha_2}{2} \sin(B_1 - B_2)\cosh(\delta_{11} - \delta_{12}),\\ G &= \frac{\alpha_1 + \alpha_2}{2} \cos(B_1 - B_2)\sinh(\delta_{11} - \delta_{12})\\ &\quad + \frac{\alpha_1 - \alpha_2}{2} \cos(B_1 + B_2)\sinh(\delta_{11} + \delta_{12}),\end{aligned} \tag{31}$$

where $B_1 = \alpha_1 x + 4\alpha_1^3 t + \delta_{21}$ and $B_2 = \alpha_2 x + 4\alpha_2^3 t + \delta_{22}$.

So the positon interaction solution of the complex KdV equation can be expressed by (16) and (30), and the solution of the coupled KdV equation can be obtained by substituting (31) into (17).

The nonsingularity condition now reads

$$\begin{aligned}(\alpha_2^2 - \alpha_1^2)\big[&\alpha_1^2 \sinh^2(\delta_{12}) - \alpha_1\alpha_2 \coth(2\delta_{11})\sinh(2\delta_{12})\\ &+ \alpha_2^2 \cosh^2(\delta_{12})\big] > 1,\end{aligned}$$

or

$$\begin{aligned}(\alpha_2^2 - \alpha_1^2)\big[&\alpha_1^2 \sinh^2(\delta_{12}) - \alpha_1\alpha_2 \coth(2\delta_{11})\sinh(2\delta_{12})\\ &+ \alpha_2^2 \cosh^2(\delta_{12})\big] < 0,\end{aligned}$$

or

$$\begin{aligned}(\alpha_1^2 - \alpha_2^2)\big[&\alpha_1^2 \cosh^2(\delta_{11}) - \alpha_1\alpha_2 \coth(2\delta_{12})\sinh(2\delta_{11})\\ &+ \alpha_2^2 \sinh^2(\delta_{11})\big] > 1,\end{aligned}$$

or

$$\begin{aligned}(\alpha_1^2 - \alpha_2^2)\big[&\alpha_1^2 \cosh^2(\delta_{11}) - \alpha_1\alpha_2 \coth(2\delta_{12})\sinh(2\delta_{11})\\ &+ \alpha_2^2 \sinh^2(\delta_{11})\big] < 0.\end{aligned}$$

According to (31), naturally, the positon interaction solution will be a periodic function if parameters α_1 and α_2 satisfy

$$\left|\frac{\alpha_1}{\alpha_2}\right| = \frac{n_1}{n_2},$$

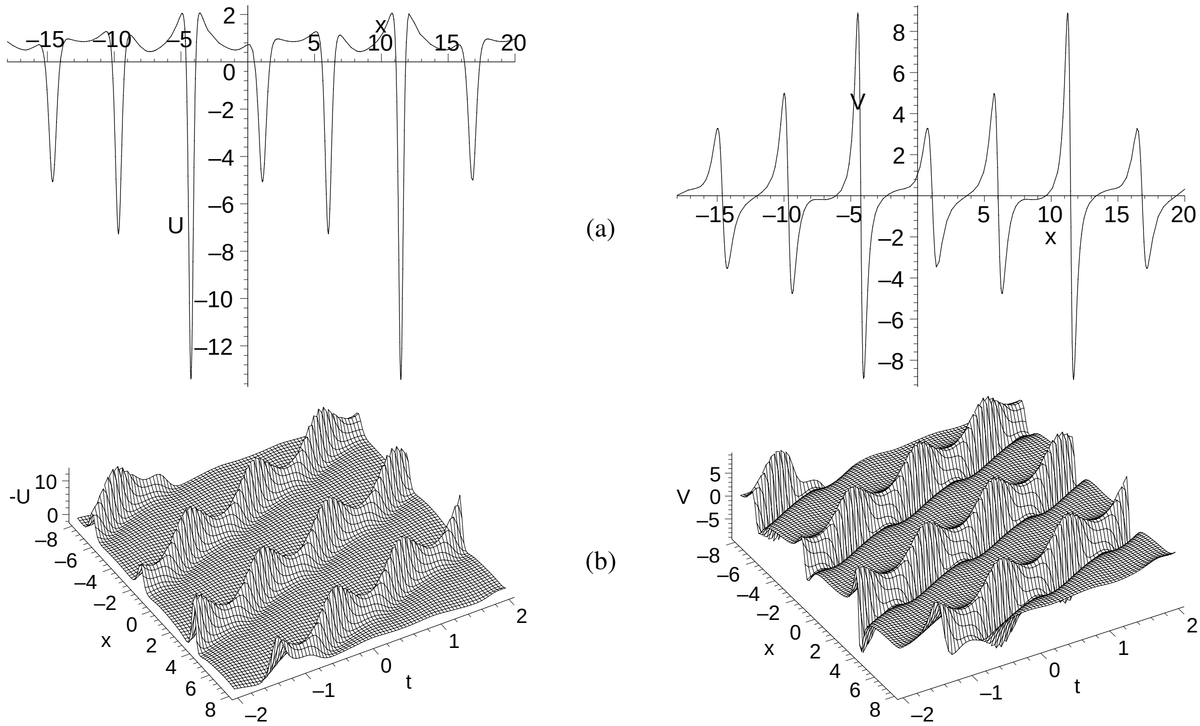

Fig. 13. Positon interaction solution (P_1-P_2-type) with the parameter choice $\alpha_1 = 1.0$, $\alpha_2 = 0.6$, $\delta_{11} = 2.0$, $\delta_{21} = 1.0$, $\delta_{12} = 1.0$, $\delta_{22} = 1.0$; (a) characteristic waveform at $t = 0$; (b) spatio-temporal evolution plot.

where n_1, n_2 are integer and without any common divisor. It is clear that the corresponding typical wavelength is

$$L = \frac{2n_1\pi}{|\alpha_1|} = \frac{2n_2\pi}{|\alpha_2|},$$

and the period is

$$T = \frac{2n_1^3\pi}{|\alpha_1^3|} = \frac{2n_2^3\pi}{|\alpha_2^3|}.$$

To classify the characteristic structure of the field U of the positon interaction solution, firstly, we select two accessorial functions as $W_1 = U|_{B_1=0}$ and $W_2 = U|_{B_2=0}$. Naturally, the field U possesses three different kinds of characteristic structures; for details, see Figs. 12, 13 and 14.

3.4. Complexiton Interaction Solutions

Generally, the solution (17) with

$$\begin{aligned} f_1 &= \cosh\left[(\mathrm{i}\alpha_1 - \beta_1)x - 4(\mathrm{i}\alpha_1 - \beta_1)^3 t + \delta_{11} + \mathrm{i}\delta_{12}\right], \\ f_2 &= \cosh\left[(\mathrm{i}\alpha_2 - \beta_2)x - 4(\mathrm{i}\alpha_2 - \beta_2)^3 t + \delta_{21} + \mathrm{i}\delta_{22}\right] \end{aligned} \tag{32}$$

is a singular two-complexiton interaction solution. Figure 15 displays a special interaction process of two singular complexitons represented by (17) and (32) with the parameter choice

$$\begin{aligned} &\beta_1 = -\alpha_1 = -\alpha_2 = \delta_{12} = \delta_{22} = 1, \\ &\beta_2 = 2, \quad \delta_{11} = -\pi, \quad \delta_{21} = 3. \end{aligned} \tag{33}$$

In a special case, the resonance of two singular complexitons may form a single nonsingular resonant complexiton. Actually, a nonsingular negaton is the usual soliton and a nonsingular resonant complexiton is the usual breather!

To demonstrate this point of view, one can take two spectral parameters as conjugate complex numbers, namely

$$\begin{aligned} \lambda_0 &= \lambda_1 + \mathrm{i}\lambda_2 = (\alpha^2 - \beta^2) + \mathrm{i}(2\alpha\beta), \\ \lambda_0' &= \lambda_1 - \mathrm{i}\lambda_2 = (\alpha^2 - \beta^2) - \mathrm{i}(2\alpha\beta) \end{aligned} \tag{34}$$

and

$$\begin{aligned} f_1 &= \cosh\left[(\mathrm{i}\alpha - \beta)x - 4(\mathrm{i}\alpha - \beta)^3 t + \delta_{11} + \mathrm{i}\delta_{12}\right], \\ f_2 &= \cosh\left[(\mathrm{i}\alpha + \beta)x - 4(\mathrm{i}\alpha + \beta)^3 t + \delta_{21} + \mathrm{i}\delta_{22}\right]. \end{aligned} \tag{35}$$

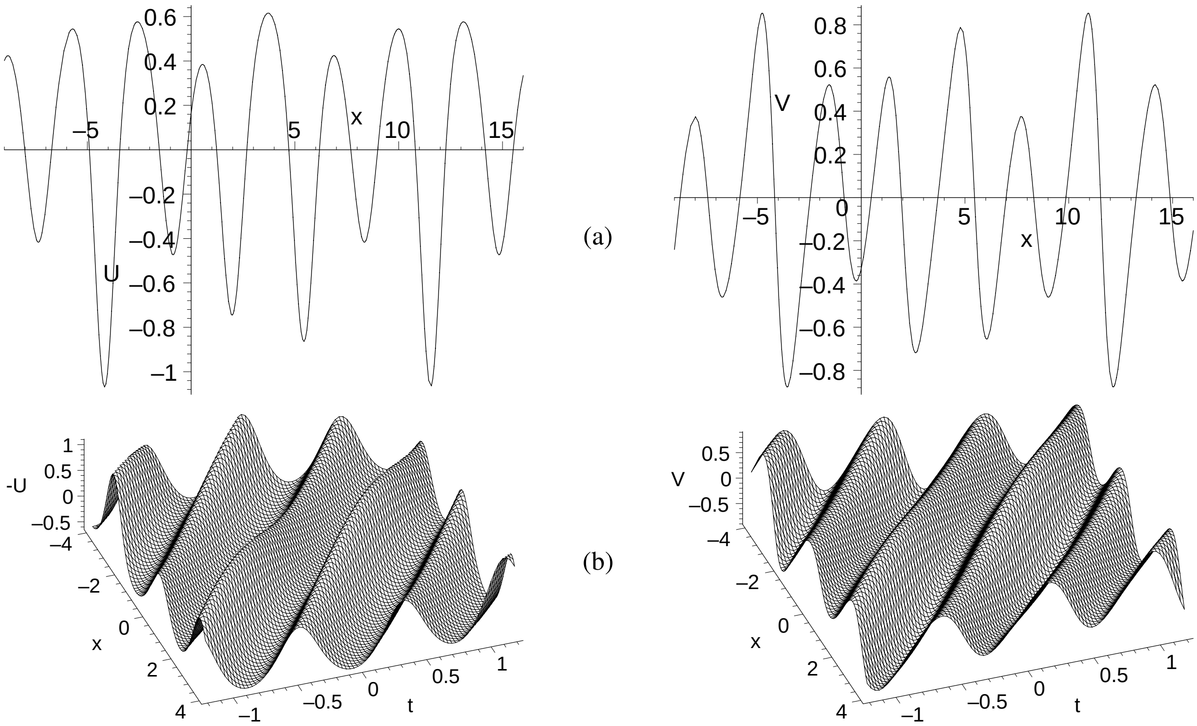

Fig. 14. Positon interaction solution (P_2-P_2-type) with the parameter choice $\alpha_1 = 1.0$, $\alpha_2 = 0.6$, $\delta_{11} = 2.0$, $\delta_{21} = 1.0$, $\delta_{12} = 2.0$, $\delta_{22} = 1.0$; (a) characteristic waveform at $t = 0$; (b) spatio-temporal evolution plot.

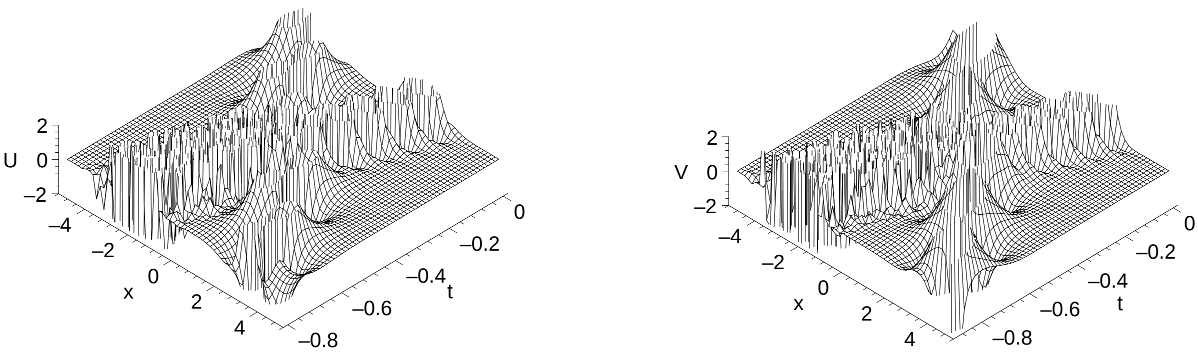

Fig. 15. Time evolution plot of a singular two-complexiton interaction solution represented by (17) and (32) with the parameter choice (33).

Substituting (35) into (15) yields

$$\begin{aligned}\psi = -\mathrm{i}\{&\alpha\sinh(2\beta x + 24\alpha^2\beta t - 8\beta^3 t \\ &- \delta_{11} + \delta_{21} - \mathrm{i}\delta_{12} + \mathrm{i}\delta_{22}) \\ &+ \beta\sin(2\alpha x - 24\alpha\beta^2 t + 8\alpha^3 t \\ &+ \delta_{12} + \delta_{22} - \mathrm{i}\delta_{11} - \mathrm{i}\delta_{21})\}.\end{aligned} \tag{36}$$

Separating the real and imaginary parts of (36), we have

$$\begin{aligned}F &= \alpha\cosh(A_1)\sin(B_1) - \beta\sinh(A_2)\cos(B_2), \\ G &= -\alpha\sinh(A_1)\cos(B_1) - \beta\cosh(A_2)\sin(B_2),\end{aligned} \tag{37}$$

where

$$\begin{aligned}A_1 &= 2x\beta + 24t\beta\alpha^2 - 8t\beta^3 - \delta_{11} + \delta_{21}, \\ A_2 &= \delta_{11} + \delta_{21}, \\ B_1 &= \delta_{22} - \delta_{12}, \\ B_2 &= 2x\alpha + 8t\alpha^3 + \delta_{22} - 24t\beta^2\alpha + \delta_{12}.\end{aligned} \tag{38}$$

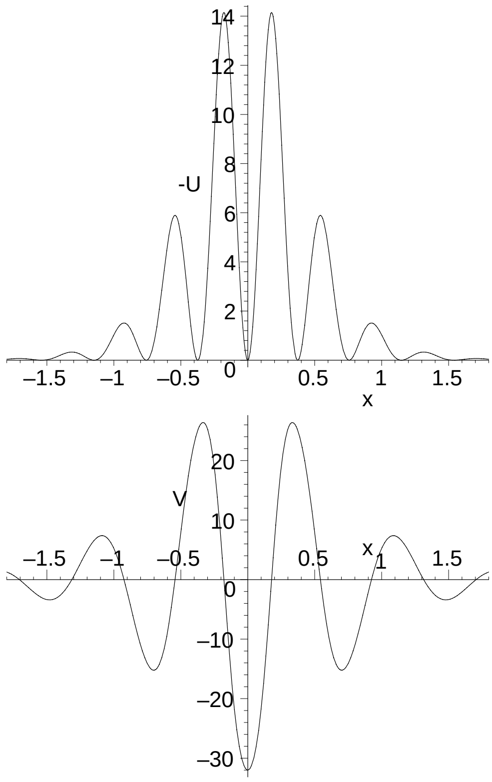

Fig. 16. A special nonsingular complexiton resonant interaction solution with the parameter choice (39) at the time $t = 0$.

So the complexiton interaction solution to the complex KdV equation can be expressed by (16) and (36) while the solution to the coupled KdV equation can be obtained by substituting (37) into (17).

It is not very difficult to find that the solution (17) with (37) is nonsingular if the parameters satisfy

$$\alpha^2 \sin^2(B_1) - \beta^2 \sinh^2(A_2) < 0, \quad \sin(2B_1) \neq 0,$$

or

$$\alpha^2 \sinh^2(A_2) - \beta^2 \sin^2(B_1) < 0.$$

It is clear that, if we take $A_2 = 0, B_1 = \pm\pi/2$, the spectral parameters α and β can be arbitrary real constants. Figure 16 displays the special nonsingular complexiton resonant interaction structure, a breather, with the parameter choice

$$\begin{aligned} &\alpha = 4, \quad \beta = 1, \\ &\delta_{11} = 0, \quad \delta_{21} = 0, \quad \delta_{12} = \pi/2, \quad \delta_{22} = 0 \end{aligned} \tag{39}$$

at the time $t = 0$, while Fig. 17 shows the detailed breather evolution process with the same parameter selections as in Figure 16.

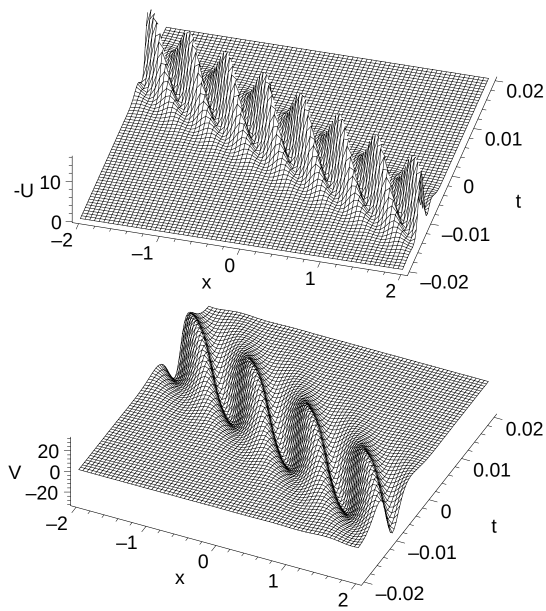

Fig. 17. Evolution process of the nonsingular complexiton resonant interaction solution with the same parameter choice as in Figure 16.

Figure 18 shows another special breather structure with the parameter selections

$$\begin{aligned} &\alpha = 2, \quad \beta = 1, \\ &\delta_{11} = \ln(2), \quad \delta_{21} = 0, \quad \delta_{12} = 0, \quad \delta_{22} = \pi/2 \end{aligned} \tag{40}$$

at $t = 0$. In the same way, Fig. 19 is a detailed time evolution plot of the breather shown in Figure 18.

4. Summary and Discussion

In this paper, various exact solutions of the general coupled Korteweg – de Vries system are obtained via the Darboux transformation of the complex KdV equation. It is found that there are two different kinds of characteristic structures with the corresponding preferences for both cases, the single negaton and positon via the first step DT.

With the help of the second step DT, one can generally obtain two singular complexiton interaction solutions. Under some suitable choices of the parameters, abundant analytical negaton-negaton, negaton-positon, positon-positon interaction solutions and resonant complexiton (breather) solutions are obtained analytically and displayed graphically.

Because the model equation system (1) can be applied in some different kinds of physical fields such

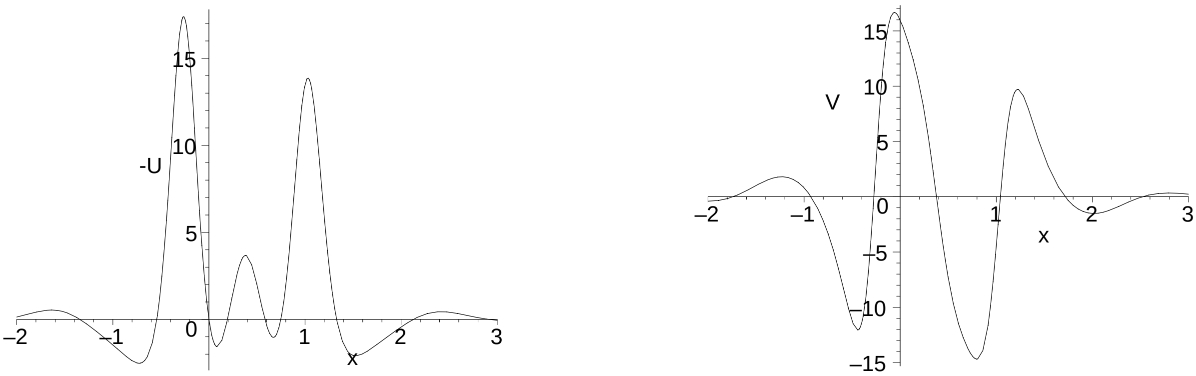

Fig. 18. A typical nonsingular complexiton resonant interaction solution with the parameter choice (40) at the time $t = 0$.

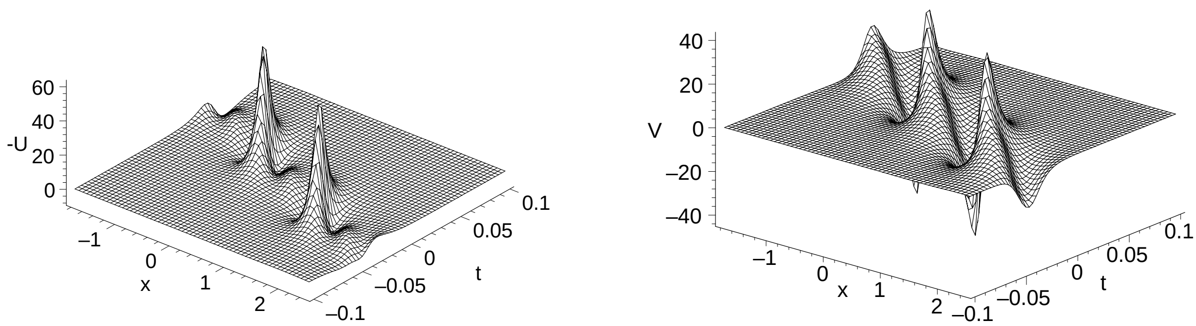

Fig. 19. Evolution process of the nonsingular complexiton resonant interaction solution with the same parameter choice as in Figure 18.

as the two-wave modes in a shallow stratified liquid [4], two-layer models of atmospheric dynamical systems [5] and two-component Bose-Einstein condensates [6], the soliton solutions obtained here may be used to describe some types of possible physical or natural phenomena. For instance, the single analytical negaton solutions may be used as some possible alternative atmospheric blocking descriptions, the analytical negaton-positon interactions may be used to describe the interactions among the background and blocking while the complexiton solution may be responsible for the blocking oscillations.

It is worth pointing out that making use of the complex KdV equation in solving the problem is more convenient than the direct use of the DTs of the coupled KdV system in [5] especially for the multiple step DTs. It is a tedious work to find the conditions for regularity when the solutions are complicated, say the multiple complexiton solutions obtained from the N-step DTs for $N \geq 3$. Possible convenient ways to obtain nonsingular multiple negaton-positon-complexiton solutions are worth to be studied.

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[1] V. B. Matveev, Phys. Lett. A **166**, 205 (1992).
[2] C. Rasinariu, J. Sukhatme, and A. Khare, J. Phys. A: Math. Gen. **29**, 1803 (1996).
[3] W. X. Ma, Phys. Lett. A **301**, 35 (2002).
[4] J. A. Gear and R. Grimshaw, Stud. Appl. Math. **70**, 235 (1984); J. A. Gear, Stud. Appl. Math. **72**, 95 (1985).
[5] S. Y. Lou, B. Tong, H. C. Hu, and X. Y. Tang, J. Phys. A: Math. Gen. **39**, 513 (2006).
[6] V. A. Brazhnyi and V. V. Konotop, Phys. Rev. E **72**, 026616 (2005).
[7] H. C. Hu, B. Tong, and S. Y. Lou, Phys. Lett. A **351**, 403 (2006).